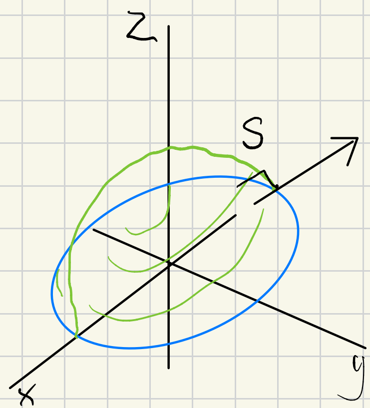
Pre-class Warm-up!!!

Would you choose to use Stokes' theorem to do the following problem?

Let F be the vector field
F(x,y,z) = (x + y^3z, sin(x) + z, y).
Compute the flux of F across the upper unit hemisphere S, that is
∬_S F · dS



a. Yes

b. No

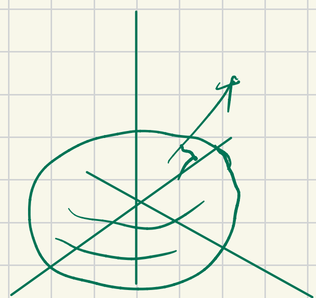
Stokes: $\iint_S \nabla \times G \cdot d\underline{S} = \int_{\partial S} G \cdot d\underline{s}$

Here F is not $\nabla \times G$ for any G
because $\nabla \cdot F = 1 + 0 + 0 = 1 \neq 0$
Stokes doesn't immediately apply.

Section 8.4: Gauss's Theorem

We learn:

- Gauss's theorem is similar to Green's theorem and Stokes' theorem, except it works one dimension higher.
- How the surface of a solid region must be oriented for Gauss's theorem
- The use of Gauss's theorem in computing the flux of a vector field across a surface.
- The proof looks very like the proof of Green's theorem
- It is also called the Divergence Theorem.



Gauss's Theorem.

Let W be a solid region of R^3 bounded by a closed surface S = ∂W oriented so that the unit normal points to the outside of the W.

If F : R^3 -> R^3 is a vector field we have

$$\iiint_W \nabla \cdot F \, dV = \iint_S F \cdot d\underline{S}$$

What do closed surface S and ∂W mean?

A closed surface has empty boundary ∂S and is bounded.

W is the region inside S.

Example. (Like most HW questions)

Find the flux of F across S where S is the unit sphere with outward pointing normal, and
F(x,y,z) = $\left(x + e^{\cos y}\sqrt{1+z^2},\ y + \frac{x}{1+z^2},\ 2z - 3y^3\right)$

Solution: Let W be the unit ball, so that $\partial W = S$. S has the correct orientation for Gauss's theorem

$$\iint_{S=\partial W} F \cdot d\underline{S} = \iiint_W \nabla \cdot F \, dV$$

$$\nabla \cdot F = 1 + 1 + 2 = 4$$

The integral is $4\iiint_W dV = 4 \cdot \frac{4\pi}{3}$

$$= \frac{16\pi}{3} \qquad \square$$

$$\text{Div } F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}$$

$$\text{“}=\text{”} \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) \cdot (F_1, F_2, F_3)$$

(x + e^cos(y) √1+z^2, y + x/(1+z^2), 2z-3y^3)

Comparison with Stokes' theorem.

Gauss

$$\iint_{\partial W} F \cdot dS = \iiint_W \nabla \cdot F \, dV$$

Stokes

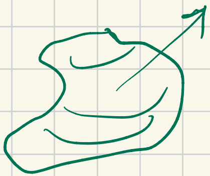
$$\int_{\partial S} F \cdot ds = \iint_S \nabla \times F \cdot dS$$

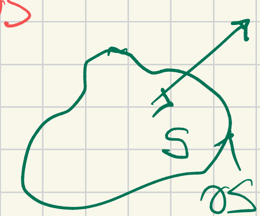
Dimension 2 3 1 2

∂ is transferred

$\partial W \leftrightarrow \nabla \cdot F$ $\quad$ $\partial S \leftrightarrow \nabla \times F$

Orientation





If Div F = 0 and S is closed we can deduce that ∬_S F·dS = 0 in two ways.

1. Use Stokes. Because $\nabla \cdot F = 0$, $F = \nabla \times G$ for some vector field G.

Stokes' $$\iint_S F \cdot dS = \iint_S \nabla \times G \cdot dS$$

$$= \int_{\partial S} G \cdot ds = 0$$ because ∂S is empty.

2. Use Gauss: $$\iint_S F \cdot dS = \iint_{\partial W} F \cdot dS$$

Let W be the region inside S.

$$= \iiint_W \nabla \cdot F \, dV = 0.$$

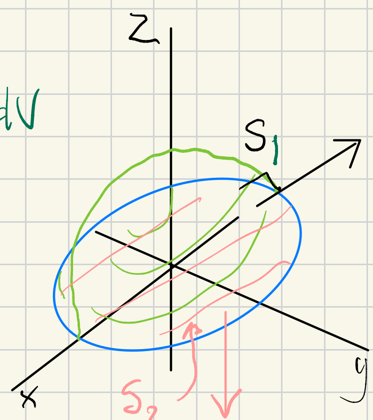
Example (closing a surface)
Let F(x,y,z) =
(x + e^cos(y) √(1+z^2), y + x/(1+z^2),
2z-3y^3)
Find the flux of F across the upper half of the unit sphere, oriented with an outward pointing normal.

Solution. Let W be the region between the upper hemisphere and the xy plane so $\partial W = S_1 \cup S_2$

Gauss says

$$\iint_{\partial W} F \cdot dS = \iiint_W \nabla \cdot F \, dV$$

$$\nabla \cdot F = 1 + 1 + 2$$
$$= 4$$



$$\iiint_W 4 \, dV = 4 \text{ volume of } W = 4 \cdot \frac{2\pi}{3} = \frac{8\pi}{3}$$

$$= \iint_{S_1} F \cdot d\underline{S} + \iint_{S_2} F \cdot d\underline{S}$$

$$\iint_{S_1} F \cdot dS = \iiint_W \nabla \cdot F \, dV - \iint_{S_2} F \cdot dS$$

On S_2 a unit normal is $(0,0,-1)$

$$F \cdot (0,0,-1) = 2z - 3y^3$$

$$\iint_{S_2} F \cdot dS = \iint_{\text{disk in } xy\text{-plane}} 2z - 3y^3 \, dx \, dy$$

$$= 0 \quad (z = 0,\ 3y^3 \text{ is an odd function})$$

$$\iint_{S_1} F \cdot dS = \frac{8\pi}{3}. \quad \square$$

Example. (Solid regions with holes in them)
Theorem 10 in the book:

The flux of $(x,y,z) / r^3$ across the surface of
an 'elementary symmetric region' M is
4π if $(0,0,0)$ lies in M
0 if $(0,0,0)$ does not lie in M.